

THE FOUR-POINT FUNCTION ON A SURFACE OF INFINITE GENUS *

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Abstract. The four-point function arising in the scattering of closed bosonic strings in their tachyonic ground state is evaluated on a surface of infinite genus. The amplitude has poles corresponding to physical intermediate states and divergences at the boundary of moduli space, but no new types of divergences result from the infinite number of handles. The implications for the universal moduli space approach to string theory are briefly discussed.

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The geometry of string perturbation theory suggests that a more complete formulation might be achieved by using the analyticity and connectedness properties of a universal moduli space [1] or Grassmannian [2] to treat the amplitudes at arbitrary genus uniformly. Both spaces contain points corresponding to infinite-genus surfaces, and it would be useful to establish if calculations of correlation functions and scattering amplitudes can be extended to this order.

The integration of the positions of vertex operators over world sheets corresponding to surfaces of infinite genus could involve unphysical divergences as a result of the non-compactness of the manifold. The four-point amplitude for closed bosonic strings is computed here when the surface is a sphere with an infinite number of handles which become infinitesimally small (Fig. 1).

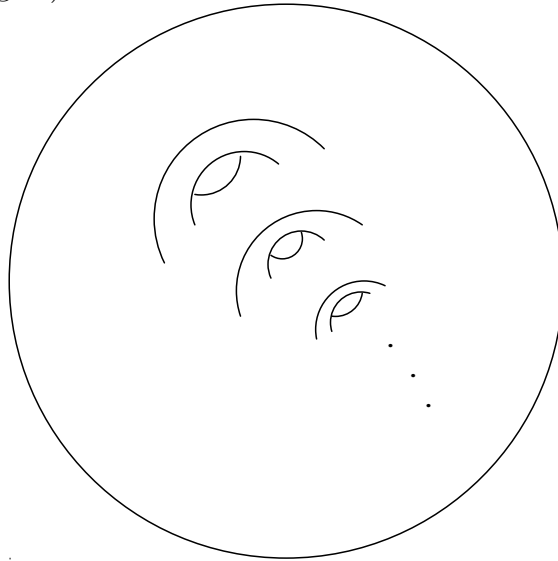


Fig. 1. A sphere with an infinite number of handles. The distance between the handles and their size decreases to zero.

It will be shown that the amplitude is finite, except for poles in the momenta, which occur at values corresponding to physical intermediate states, and divergences associated with the boundary of moduli space. The result depends on the rate at which spacing between the handles and their size decreases to zero.

Since the scattering amplitudes for closed bosonic strings depends on the correlation functions of vertex operators, the Green function for the scalar Laplacian on a Riemann surface is required when the incoming and outgoing strings are in their tachyonic ground states. Although the Green function on a compact surface may be expressed in terms of prime forms, it also can be obtained by using the representation, up to conformal equivalence of the surface as D/Γ , where D is a domain in the extended complex plane and Γ is a discontinuous subgroup of $PSL(2, \mathbb{C})$ leaving D invariant [3], and then applying the method of images. In particular, uniformization of a closed surface of genus g by a Schottky group generated by the linear transformations T_1, \dots, T_g leads to an expression for the Green function with two sources located at the points z_R and z_S

$$\begin{aligned}
G_{QS}(P, R) &= \sum_{\alpha} \ln \left| \frac{z_P - V_{\alpha} z_R}{z_P - V_{\alpha} z_S} \frac{z_Q - V_{\alpha} z_S}{z_Q - V_{\alpha} z_R} \right| \\
&\quad - \frac{1}{2\pi} \sum_{m,n=1}^g \operatorname{Re}\{v_m(z_P) - v_m(z_Q)\} (Im \tau)_{mn}^{-1} \operatorname{Re}\{v_n(z_R) - v_n(z_S)\} \\
v_n(z) &= \sum_{\alpha} {}^{(n)} \ln \left(\frac{z - V_{\alpha} \xi_{1n}}{z - V_{\alpha} \xi_{2n}} \right) \quad v_n(z) - v_n(T_m z) = 2\pi i \tau_{mn}
\end{aligned} \tag{1}$$

where the V_{α} are arbitrary products of the generators T_1, \dots, T_g , ξ_{1n}, ξ_{2n} are the two fixed points of T_n , $\sum_{\alpha} {}^{(n)}$ represents the sum over all V_{α} that do not have $T_n^{\pm 1}$ at the right-hand end of the product, and τ is the period matrix of the surface. The finiteness of the sums in equation (1) depends on the convergence of the Poincare series $\sum_{\alpha \neq I} |\gamma_{\alpha}|^{-2}$ [4], where $V_{\alpha} z = \frac{\alpha_{\alpha} z + \beta_{\alpha}}{\gamma_{\alpha} z + \delta_{\alpha}}$.

For the extension of the Schottky group to an infinite number of generators, the isometric circles $I_{T_n} = \{z \in \hat{\mathbb{C}} | |\gamma_n z + \delta_n| = 1\}$ can be joined to $I_{T_n^{-1}} = \{z \in \hat{\mathbb{C}} | |\gamma_n z - \alpha_n| = 1\}$ for all n to create a sphere with an infinite number of handles [5]. While the expression for the Green function in terms of prime forms is no longer available, because the theta function has been generalized only for particular infinite-genus surfaces [6], the method of images can be used again to obtain a Green function in the form (1).

A new proof of convergence of the Poincare series is necessary when the uniformizing group Γ is infinitely generated. Since $\sum_{\alpha \neq I} |\gamma_{\alpha}|^{-2} = \sum_{\alpha \neq I} r_{\alpha}^2$, where r_{α} is the radius of

I_{V_α} , the sum can be shown to be finite when r_n decreases to zero sufficiently fast as $n \rightarrow \infty$ [7]. For instance, suppose that the distances between the isometric circles are bounded below, and that the distance between I_{T_n} and $I_{T_n^{-1}}$ is bounded above for all n , so that the circles accumulate at ∞ as in Fig. 2. Then, if $r_n = kn^{-2}$, with k being a constant that depends on the spacing between the circles, the series converges. This result can be demonstrated by grouping the elements of Γ according to the number of fundamental generators in the product. Let $V_{(l)}$ be an element of the Schottky group at level l consisting of the product of l fundamental generators. Suppose that $V_{(l+1)} = T_n V_{(l)}$. Then

$$\left| \frac{\gamma_{(l+1)}}{\gamma_{(l)}} \right| > (|K_n|^{-\frac{1}{2}} - |K_n|^{\frac{1}{2}}) \left| \frac{\xi_{2n} - \frac{\alpha_{(l)}}{\gamma_{(l)}}}{\xi_{2n} - \xi_{1n}} \right| - |K_n|^{\frac{1}{2}} \quad (2)$$

If the decrease of the absolute values of the multipliers is given by $|K_n|^{\frac{1}{2}} = (c_1 n^2 + c_2)^{-1}$, the ratio in equation (2) is greater than $c_1 c n^2$ when $c_2^2 > 1 + \frac{1}{c}$, where c is the lower bound for $\left| \frac{\xi_{2n} - \frac{\alpha_{(l)}}{\gamma_{(l)}}}{\xi_{2n} - \xi_{1n}} \right|$. Denoting the upper bound for $|\xi_{2n} - \xi_{1n}|$ by c' , it follows that $|\gamma_n|^{-2} < \frac{c'^2}{c_1^2 n^4}$ and

$$\sum_{\alpha \neq I} |\gamma_\alpha|^{-2} < c^2 c'^2 \left[\frac{2}{c_1^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^4} + \left(\frac{2}{c_1^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^4} \right)^2 + \dots \right] \quad (3)$$

which converges when $c_1^2 > \frac{2}{c^2} \sum_{n=1}^{\infty} \frac{1}{n^4}$. In terms of the radii of the isometric circles of the fundamental generators, the decrease is given by kn^{-2} , where $k < \frac{c'}{c_1}$.

While convergence of the Poincare series implies finiteness of the first sum in equation (1), the second term becomes

$-\frac{1}{2\pi} \sum_{m,n=1}^{\infty} \text{Re}\{v_m(z_P) - v_m(z_Q)\} (\text{Im } \tau)_{mn}^{-1} \text{Re}\{v_n(z_R) - v_n(z_S)\}$ at $g = \infty$. This sum is well-defined when the imaginary part of the period matrix is positive-definite, a property which follows from the bilinear relations for harmonic differentials on a Riemann surface of class O_G [8][9]. In particular, the inverse of the period matrix exists for the class of surfaces considered above. From the formula relating the entries of the period matrix and the Schottky group parameters

$$\tau_{mn} = \frac{1}{2\pi i} \left[\ln K_m \delta_{mn} + \sum_{\alpha} {}^{(m,n)} l_n \left(\frac{\xi_{1m} - V_\alpha \xi_{1n}}{\xi_{1m} - V_\alpha \xi_{2n}} \frac{\xi_{2m} - V_\alpha \xi_{2n}}{\xi_{2m} - V_\alpha \xi_{1n}} \right) \right] \quad (4)$$

where the elements V_α having $T_m^{\pm 1}$ as the left-most member and $T_n^{\pm 1}$ as right-most member of the product are excluded from the summation, it follows that

$$Im \tau_{nn} \simeq \frac{2}{\pi} \ln n + \frac{1}{\pi} \ln c_1 + \frac{c_2}{\pi c_1 n^2} \quad (5)$$

Both the off-diagonal elements of $Im \tau$ and the functions $Re v_n(z)$ can be estimated simultaneously. Consider a point z at a bounded distance $d(z, I_{T_{n_0}})$ from $I_{T_{n_0}}$ for finite n_0 . Since

$$Re v_n(z) = \sum_{\alpha} {}^{(n)} \ln \left| 1 + \frac{(\xi_{2n} - \xi_{1n})\gamma_{\alpha}^{-2}}{(\xi_{1n} + \frac{\delta_{\alpha}}{\gamma_{\alpha}})(\xi_{2n} + \frac{\delta_{\alpha}}{\gamma_{\alpha}})(z - V_{\alpha}\xi_{2n})} \right| \equiv \sum_{\alpha} {}^{(n)} Re v_{n\alpha}(z) \quad (6)$$

the elements V_{α} can be separated into the following categories:

$$\begin{aligned} (i) \quad & d(I_{V_{\alpha}}, I_{T_{n_0}}), d(I_{V_{\alpha}^{-1}}, I_{T_{n_0}}) \text{ are bounded} \\ & Re v_{n\alpha}(z) = O\left(\frac{1}{n^2}\right) |\gamma_{\alpha}|^{-2} \\ (ii) \quad & d(I_{V_{\alpha}}, I_{T_{n_0}}), d(I_{V_{\alpha}^{-1}}, I_{T_n}) \text{ are bounded} \\ & Re v_{n\alpha}(z) = O\left(\frac{1}{n^3}\right) |\gamma_{\alpha}|^{-2} \\ (iii) \quad & d(I_{V_{\alpha}}, I_{T_n}), d(I_{V_{\alpha}^{-1}}, I_{T_{n_0}}) \text{ are bounded} \\ & Re v_{n\alpha}(z) = O(1) |\gamma_{\alpha}|^{-2} \\ (iv) \quad & d(I_{V_{\alpha}}, I_{T_n}), d(I_{V_{\alpha}^{-1}}, I_{T_n}) \text{ are bounded} \\ & Re v_{n\alpha}(z) = O\left(\frac{1}{n}\right) |\gamma_{\alpha}|^{-2} \end{aligned} \quad (7)$$

The sum over the elements in the categories (iii) and (iv) may be estimated by repeating the analysis of the Poincare series $\sum_{\alpha \neq I} |\gamma_{\alpha}|^{-2}$ with the restriction $I_{V_{\alpha}} \subset D_{T_m}, D_{T_m^{-1}}$ for $m \geq n$, where D_{T_m} is the isometric disk $D_{T_m} = \{z \in \hat{\mathbb{C}} \mid |\gamma_m z + \delta_m| \leq 1\}$. The sum can be bounded by

$$2 \frac{c'^2}{c_1^2} \frac{\zeta(4, n)}{1 - \zeta(4, n)} \quad (8)$$

with $\zeta(4, n)$ being the generalized zeta function. It may be verified that a lower bound with similar dependence on n can be placed on the restricted Poincare series. Using Hermite's representation of the generalized zeta function, one observes that the bound (8) decreases

as $\frac{2c'^2}{3c_1^2} \frac{1}{n^3}$. Since the contributions of the elements in categories (i) and (ii) decrease as $O(\frac{1}{n^2})$ and $O(\frac{1}{n^3})$ respectively, $Re v_n(z) = O(\frac{1}{n^2})$. Thus, $Re \{v_n(z_P) - v_n(z_Q)\} < \frac{v_{PQ}}{n^2}$ for an appropriate constant v_{PQ} . The fall-offs of the entries $(Im \tau)_{mn}$, $m \neq n$ are then

$$Im \tau_{mn} = O\left(\frac{1}{|m - n|^2}\right) \quad (9)$$

Since diagonalization of the matrix $(Im \tau)^{-1}$ produces eigenvalues $\lambda_n = \frac{\pi}{2} \frac{1}{\ln n}$ for large n , $\frac{1}{2\pi} \sum_{m,n=1}^{\infty} Re \{v_m(z_P) - v_m(z_Q)\} (Im \tau)_{mn}^{-1} Re \{v_n(z_R) - v_n(z_S)\} < \frac{1}{4} v_{PQ} v_{RS} \sum_{n=1}^{\infty} \frac{1}{n^4 \ln n}$ so that this term is also finite confirming the results of a general proof given in earlier work [7].

Having established the suitability of the series expansion (1) for the Green function, one would like to determine its behaviour near the isometric circles, particularly in the region where they accumulate at ∞ , because the scattering amplitude for N tachyons

$$\begin{aligned} & \sum_{g=0}^{\infty} \kappa^g \int_{M_g} d\mu_g \int_{\Sigma_g} \prod_{i=1}^N d^2 z_i \sqrt{g(z_i)} \prod_{i < j} e^{-\frac{p_i \cdot p_j}{2}} \tilde{G}^{sym}(z_i, z_j) \\ \tilde{G}^{sym}(z_i, z_j) &= G^{sym}(z_i, z_j) - \frac{1}{2} \lim_{z'_i \rightarrow z_i} [G(z_i, z'_i) - \ln d(z_i, z'_i)] \\ & \quad - \frac{1}{2} \lim_{z'_j \rightarrow z_j} [G(z_j, z'_j) - \ln d(z_j, z'_j)] \end{aligned} \quad (10)$$

involves an integration over the entire fundamental domain, Δ of the Schottky group, whose points are in one-to-one correspondence with the Riemann surface.

Consider, for example, the four-point amplitude with the positions of three of the vertex operators fixed at z_1^0, z_2^0, z_3^0

$$\begin{aligned} & f(z_1^0, z_2^0, z_3^0) \int_{\Delta} d^2 z_4 |z_4 - z_1^0|^{-\frac{p_1 \cdot p_4}{2}} \prod_{\alpha \neq I} \left| \frac{z_4 - V_{\alpha} z_1^0}{z_4 - V_{\alpha} z_4} \frac{z_1^0 - V_{\alpha} z_4}{z_1^0 - V_{\alpha} z_1^0} \right|^{-\frac{p_1 \cdot p_4}{4}} \\ & \cdot \prod_{m,n} \exp \left[-\frac{p_1 \cdot p_4}{8\pi} Re(v_m(z_4) - v_m(z_1^0)) (Im \tau)_{mn}^{-1} Re(v_n(z_4) - v_n(z_1^0)) \right] \\ & \cdot (\text{similar factors with } z_1^0 \rightarrow z_2^0, z_3^0, p_1 \rightarrow p_2, p_3) \end{aligned} \quad (11)$$

where

$$\begin{aligned}
f(z_1^0, z_2^0, z_3^0) &= |z_2^0 - z_1^0|^{2 - \frac{p_1 \cdot p_2}{2}} \prod_{\alpha \neq I} \left| \frac{z_2^0 - V_\alpha z_1^0}{z_2^0 - V_\alpha z_2^0} \frac{z_1^0 - V_\alpha z_2^0}{z_1^0 - V_\alpha z_1^0} \right|^{-\frac{p_1 \cdot p_2}{4}} \\
&\times \prod_{m,n} \exp\left[\left(-\frac{p_1 \cdot p_2}{8\pi} \operatorname{Re}(v_m(z_2^0) - v_m(z_1^0)) (Im\tau)_{mn}^{-1} \operatorname{Re}(v_n(z_2^0) - v_n(z_1^0))\right)\right] \\
&\times (\text{similar factors involving permutations of } z_1^0, z_2^0, z_3^0)
\end{aligned} \tag{12}$$

The integral (11) is finite in the neighbourhood of z_1^0, z_2^0, z_3^0 if $p_1 \cdot p_4, p_2 \cdot p_4, p_3 \cdot p_4 < 4$, while momentum conservation implies that $p_1 \cdot p_4 + p_2 \cdot p_4, p_1 \cdot p_4 + p_3 \cdot p_4, p_2 \cdot p_4 + p_3 \cdot p_4 > 4$. This demonstrates the existence of a range of momenta for which the integral is finite, allowing for analytic continuation of the amplitude to physical values of the momenta.

The integral in the neighbourhood of the isometric circles can be studied to ascertain whether the allowed range is narrowed even further. Specifically, the integral is finite in the asymptotic regions $z_4 \rightarrow -\frac{\delta_\alpha}{\gamma_\alpha}, z_4 \rightarrow V_\alpha z_1^0, z_4 \rightarrow V_\alpha z_4$, and $z_4 \rightarrow \infty$, by the property of momentum conservation. The product terms in (11) conceivably could diverge if $V_\alpha z_1^0$ or $V_\alpha z_4$ is arbitrarily close to one of the circles I_{T_n} . This does not happen, however, because, there are no limit points on any of the isometric circles $\{I_{T_n}\}$ [7]. Moreover, for the configuration in Fig. 2, if z is a bounded distance away from I_{T_n} , the maximum distance between $V_\alpha z$ and the center of $I_{T_n^{-1}}$ is $\frac{c_0 r_n}{n}$ for some constant c_0 .

This result can be obtained by considering the positions of the isometric circles $I_{(T_n T_m)}^{-1}$ inside the disk $D_{T_n^{-1}}$. Suppose that z lies on the curve defined by the equation

$$\frac{|\gamma_m|}{|\gamma_n|} \frac{|z - \frac{\alpha_m}{\gamma_m}|}{|z + \frac{\delta_n}{\gamma_n}|} = 1 \tag{13}$$

Then $T_n z$ lies on $I_{(T_n T_m)^{-1}}$ and its distance from the center of $D_{T_n^{-1}}$ is

$$\left| T_n z - \frac{\alpha_n}{\gamma_n} \right| = \frac{|\gamma_n|^{-2}}{\left| z + \frac{\delta_n}{\gamma_n} \right|} = \frac{|\gamma_n|^{-1} |\gamma_m|^{-1}}{\left| z - \frac{\alpha_m}{\gamma_m} \right|} \tag{14}$$

For $m \ll n$,

$$d(I_{(T_n T_m)^{-1}}, I_{T_n^{-1}}) \simeq |\gamma_n|^{-1} \left| 1 - \frac{1}{m^2 n} \right| \tag{15}$$

It follows that

$$|z_4 - V_\alpha z_1^0| > |\gamma_n|^{-1} \left(1 - \frac{c_0}{n}\right) \quad (16)$$

for some value of c_0 . Similarly, $|z_4 - V_\alpha z_4| < |\gamma_n|^{-1} \left(1 + \frac{c_0}{n}\right)$ when z_4 lies in an infinitesimal neighbourhood of $I_{T_n^{-1}}$.

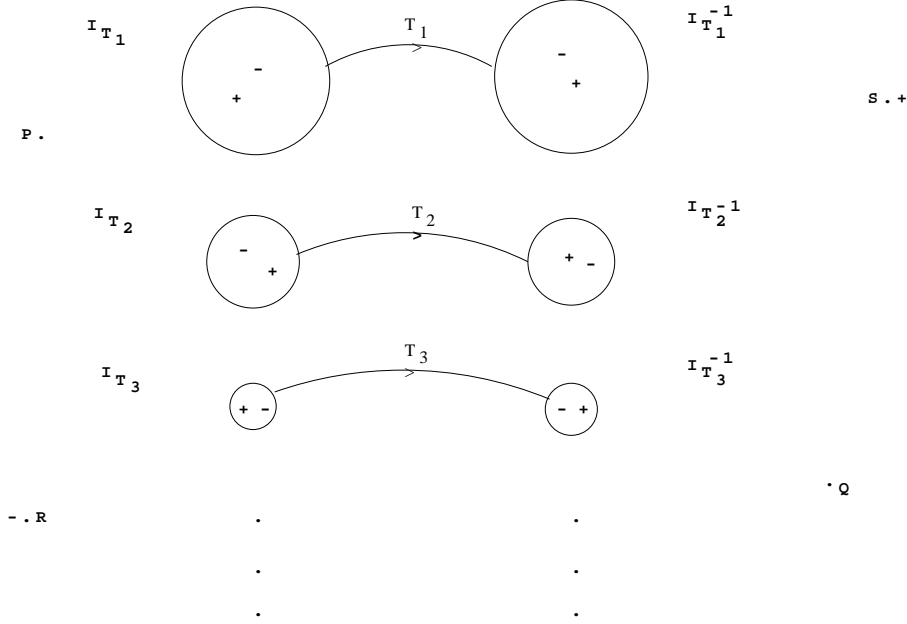


Fig. 2. The fundamental region for an infinitely discontinuous subgroup of $\text{PSL}(2, \mathbb{C})$ is the exterior of the isometric circles.

One can then place bounds such as $\frac{1 - \frac{c_0}{n}}{1 + \frac{c_0}{n}} < \left| \frac{z_4 - V_\alpha z_1^0}{z_4 - V_\alpha z_4} \right| < \frac{1 + \frac{c_0}{n}}{1 - \frac{c_0}{n}}$, which imply that the product factors in (11) tend to 1 as z_4 approaches ∞ along the direction of the isometric circles. Since $-2\frac{c_0}{n} - O(\frac{1}{n^2}) \leq \ln \left| \frac{z - V_\alpha \xi_{1m}}{z - V_\alpha \xi_{2m}} \right| \leq 2\frac{c_0}{n} + O(\frac{1}{n^2})$ when $V_\alpha z$ lies in D_{T_n} or $D_{T_n^{-1}}$, the contribution of the isometric circles accumulating at ∞ to $\text{Re } v_n(z)$ as $z \rightarrow \infty$ vanishes. As the contribution of the elements V_α , such that $I_{V_\alpha^{-1}} \subset D_{T_{n_0}}, D_{T_{n_0}^{-1}}$ for bounded n_0 also vanishes as $z \rightarrow \infty$, $\text{Re } v_n(z) \rightarrow 0$ in this limit. Therefore, since $\sum_i p_i = 0$, $p_4^2 = -8$, the integrand falls off as $|z_4|^{-4}$, and the integral is finite. For any value of s, t, u , the only singularities in the integrand occur at z_1^0, z_2^0, z_3^0 so that (11) is of the form

$$f(z_1^0, z_2^0, z_3^0) \int_{\Delta} d^2 z_4 |z_4 - z_1^0|^{-\frac{p_1 \cdot p_4}{2}} |z_4 - z_2^0|^{-\frac{p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{-\frac{p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4) \quad (17)$$

where $\Phi(z_4, \bar{z}_4)$ is regular throughout the fundamental region. By dividing the integration region into three disks of radius Λ about z_1^0 , z_2^0 , z_3^0 and the remainder of the fundamental domain, the integrand can be expanded in a Laurent series [7][10] and the integral can be shown to be equal to

$$2\pi \sum_{n=0}^{\infty} \frac{\Lambda^{-\frac{p_1 \cdot p_4}{2} + 2n + 2}}{-\frac{1}{2}p_1 \cdot p_4 + 2n + 2} \frac{1}{(n!)^2} (\partial \bar{\partial})^n \{ |z_4 - z_2^0|^{-\frac{p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{-\frac{p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4) \}_{z_4=z_1^0} \\ + (\text{similar terms with } z_1^0 \rightarrow z_2^0, z_3^0, p_1 \rightarrow p_2, p_3) + \text{finite} \quad (18)$$

In terms of the Mandelstam variables, there are simple poles at $s, t, u = 8(n-1)$, $n = 0, 1, 2, \dots$, corresponding to the tachyon and excited intermediate states.

Although the positions of three of the vertex operators are fixed in (11), the full amplitude requires an integration over z_1^0 , z_2^0 , z_3^0 also. The analysis above still holds, except that there are singularities in the product terms when z_i approaches I_{T_n} and $I_{T_n^{-1}}$ for any n . These divergences are physical, however, because z_i and z_j are approaching the same point on the world sheet.

The total scattering amplitude, given by a further integration over moduli space, would also contain divergences associated with the boundary of moduli space. For the class of surfaces defined by $|\xi_{1n} - \xi_{2n}|$ being bounded and $|K_n| = (c_1 n^2 + c_2)^{-2}$, there is a divergence in the moduli space integral given by

$$\frac{\sinh^4 \left(\sqrt{\frac{c_2}{c_1}} \pi \right)}{\pi^4 c_2^2} \lim_{g \rightarrow \infty} \frac{c_1^{4g+2} \pi^{14g-4}}{2^{13g}} (g!)^8 \prod_{n=1}^g \frac{1}{(ln \ n)^{13}} \quad (19)$$

up to exponential factors associated with products over conjugacy classes of primitive elements and fixed-point integrals [11][12].

This investigation has demonstrated that these are the only types of divergences that arise in the amplitude even though the surfaces have an infinite number of handles. This result could be extended to the scattering of other string states such as the graviton. The derivatives in the corresponding vertex operator would be expected to lead to an even faster fall-off for the correlation functions as the positions tend to infinity along the

direction of the isometric circles. Finiteness of the integral of the correlation function over the string worldsheet, except for coincidence of the vertex operators, would follow from this asymptotic behaviour.

The growth of the moduli space integral given in equation (19) is similar to the dependence of the partition function, regularized in a genus-independent manner by introducing cut-offs on the lengths of closed geodesics on the Riemann surface [13]. However, the divergence in (19) represents an independent contribution to the moduli space integral, which would be eliminated by the regularization, because there exist closed geodesics, homotopically equivalent to the A_n -cycles, or I_{T_n} on the covering surface, that have length in the intrinsic metric decreasing to zero as $n \rightarrow \infty$ [7][14].

Divergences at the boundary of moduli space can be eliminated in superstring theory at each finite order of the perturbation expansion [15]. Since the amplitudes at each order have been demonstrated to be finite, no regularization removing a neighbourhood of the boundary of moduli space is required. Effectively closed surfaces of the type considered in this paper could be included in the path integral representing the scattering amplitude. The large-order divergences found for bosonic strings may be eliminated for superstrings, since it has been shown that they arise in the Schottky group parametrization in the limits $|K_n| \rightarrow 0$ and $|\xi_{1n} - \xi_{2n}| \rightarrow 0$ [14]. However, as a larger class of surfaces is being included in the superstring path integral, the counting of the different types of surfaces would affect finiteness of the entire scattering amplitude.

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